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Ruijsenaars–Macdonald-type difference operators from Z_n Belavin model with open boundary conditions

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Abstract. Antisymmetric fusions of the Z_n Belavin model with integrable open boundary conditions are studied. The centre of the corresponding operator algebra, the Sklyanin determinant, is constructed. We prove that the transfer matrices of the fusion hierarchies mutually commute. Utilizing the factorized L operators, this commuting family is expressed as mutually commuting difference operators acting on a function space. This gives Ruijsenaars–Macdonald-type commuting difference operators, which describe the conservation laws of an integrable system.

1. Introduction

In two-dimensional classical statistical mechanics, the commuting transfer matrices play an important role because they ensure the exact solvability of the system. For periodic boundary condition, given an R-matrix which is a solution of the Yang–Baxter equation (YBE) [1, 2], and an L operator which naturally arises as the row-to-row transfer matrix, one can derive the commutativity of the transfer matrices by taking the trace in the fundamental relation $RLL = LLR$ [3, 4]. Recently, there has been increasing interest in the study of two-dimensional statistical models or integrable quantum field theory with open boundary conditions [5–12]. With open boundary conditions, besides the YBE one has to use the reflection equation (RE) [13, 14] (or boundary YBE) to obtain the commuting family of transfer matrices. In studying two-dimensional integrable models, the so-called fusion procedure has often been used to generate some new integrable models based on the elementary model [15–17]. The corresponding fused transfer matrices of the fusion hierarchies also form a one-parameter commutative family.

The Z_n Belavin model [18, 19] is a generalization of Baxter's eight-vertex model. Its reflection matrix K^- and dual reflection matrix K^+ with integrable boundary conditions were given in [9, 20]. Inspired by Bazhanov *et al* [21] and using the intertwiner of the Z_n Belavin model and along with the $A_n^{(1)}$ face model given by Jimbo *et al* [22, 23], the factorized $L(u)$ operator for the Z_n symmetric R-matrix was constructed [24–26]. Utilizing this factorized $L(u)$ operator, we can express the hierarchies of the commuting transfer matrix of the antisymmetric fused model as difference operators; these may also be considered as Ruijsenaars–Macdonald-type operators [27–30]. One of these operators can be chosen as the quantum Hamiltonian of an integrable dynamic system. Our work is an extension of Hasegawa's work in [31]. Although, because of the complication of the reflection matrix K^- and dual reflection matrix K^+, the result we obtained is not reduced to the simple form as Hasegawa did for the periodic boundary condition. However, these difference operators can indeed be related to an integrable dynamic system.

An integrable n-dimensional dynamic system has n independent quantities including the Hamiltonian which are in involution. Recently, the well known integrable Calogero–Sutherland–Moser-type models have become revitalized (e.g. see [29–37] and references therein). One of them is the Ruijsenaars–Scheider (RS) model [30] which can be related to the Z_n Belavin model [31] and elliptic quantum groups [33] in two-dimensional statistics. The quantum Hamiltonian of the RS model can be seen as one of the transfer matrices of a commuting family $\{t_m\}$, the members of which act on a function space as difference operators. In this construction, the boundary conditions are chosen to be periodic. In this paper, we antisymmetrically fuse the Z_n Belavin model with an open boundary and relate the transfer matrices of the fused model to Ruijsenaars–Macdonald-type difference operators. In a similar way to that used in [31] and [33], our hierarchy of the commuting family $\{t_m\}$ and the Ruijsenaars–Macdonald difference operators of [27–30] also describe the symmetry of an integrable system. In fact, we can choose one of these operators as the Hamiltonian of the system. The approach used is different from that in [34], where Hikami gave different RS type models from open boundaries.

The outline of the paper is as follows. In section 2 we introduce the Z_n Belavin model, its reflection matrix in the open boundary condition and some notation used in this paper. In section 3 the fusions of the R-matrices, K-matrices and T operators are presented, and the algebraic centre of the operator algebra $\mathcal{T}(u)$ is constructed. In section 4, we prove that the transfer matrices of the antisymmetric fused Z_n Belavin model form a commuting family. In section 5, after reviewing the factorized L operators, we express the fused transfer matrices in section 5 as difference operators acting on a function space, which describe the symmetry of the integrable system given in section 6, concluding with a short discussion in section 7.

2. Z_n Belavin model with open boundary conditions

In two-dimensional statistical mechanics, the Z_n Belavin model is an exactly solvable lattice model with Z_n symmetry. Let g and h be $n\times n$ matrices with elements $g_{ij}=\omega^i\delta_{ij}$ and $h_{ij}=\delta_{i+1,j}$ respectively, here $i,j\in Z_n$, $(n>1)$, and $\omega=\exp(\frac{2\mathrm{i}\pi}{n})$. For a pair of given integers, $\alpha=(\alpha_1,\alpha_2)\in G_n=Z_n\otimes Z_n$, let us define $I_\alpha=I_{\alpha_1\alpha_2}=g^{\alpha_2}h^{\alpha_1}$. Consider a space $V=V_1\otimes V_2\otimes\cdots\otimes V_m$ and $I_\alpha^{(j)}=I\otimes\cdots\otimes I_\alpha\otimes\cdots\otimes I$, I_α is at the jth site. Then the Z_n symmetric Belavin R-matrix is

$$R_{jk}(u)=\sum_{\alpha\in G_n}W_\alpha(u)I_\alpha^{(j)}\otimes(I_\alpha^{-1})^{(k)}. \tag{1}$$

Here

$$W_\alpha(u)\equiv\frac{1}{n}\frac{\sigma_\alpha(u+\frac{w}{n})}{\sigma_\alpha(\frac{w}{n})}\qquad \sigma_\alpha(u)=\theta\begin{bmatrix}\frac{1}{2}+\frac{\alpha_1}{n}\\ \frac{1}{2}+\frac{\alpha_2}{n}\end{bmatrix}(u,\tau)$$

and w and τ are fixed parameters of the model. The Jacobi theta function $\theta\begin{bmatrix}a\\ b\end{bmatrix}(u,\tau)$ is characterized by $a,b\in R$:

$$\theta\begin{bmatrix}a\\ b\end{bmatrix}(u,\tau)=\sum_{m\in Z}\exp\{\mathrm{i}\pi(m+a)^2\tau+2\mathrm{i}\pi(m+a)(u+b)\}.$$

This R-matrix is $Z_n \otimes Z_n$ symmetric in the sense that

$$I_\alpha^{(j)} \otimes I_\alpha^{(k)} R_{jk}(u) (I_\alpha^{-1})^{(j)} \otimes (I_\alpha^{-1})^{(k)} = R_{jk}(u). \tag{2}$$

It also satisfies the YBE:

$$R_{12}(u_1 - u_2) R_{13}(u_1) R_{23}(u_2) = R_{23}(u_2) R_{13}(u_1) R_{12}(u_1 - u_2) \tag{3}$$

initial condition

$$R_{12}(0) = P_{12} \tag{4}$$

unitarity

$$R_{12}(u) R_{21}(-u) = \rho(u)\mathrm{id} \tag{5}$$

and cross-unitarity

$$R_{12}^{t_1}(u) R_{21}^{t_1}(-u - nw) = \tilde{\rho}(u)\mathrm{id} \tag{6}$$

where

$$\rho(u) = \frac{\sigma_0(u+w)\sigma_0(u-w)}{\sigma_0^2(w)} \qquad \tilde{\rho}(u) = \frac{\sigma_0(u)\sigma_0(-u-nw)}{\sigma_0^2(w)}$$

t_i denotes transposition in the ith space, P_{jk} is the permutation operator acting on the jth and kth spaces. In the framework of the quantum inverse-scattering method, there is an $\overset{j}{L}_h(u)$ operator which is defined on the tensor product space of the jth auxiliary space V_j and the local quantum space h. It satisfies

$$R_{12}(u_1 - u_2) \overset{1}{L}_h(u_1) \overset{2}{L}_h(u_2) = \overset{2}{L}_h(u_2) \overset{1}{L}_h(u_1) R_{12}(u_1 - u_2). \tag{7}$$

Utilizing equation (7) repeatedly, we can have the row-to-row monodromy matrix $\overset{j}{T}(u)$ defined on $V_j \otimes (h_1 \otimes \cdots \otimes h_N)$:

$$T_j(u) = \overset{j}{L}_{h_N}(u) \ldots \overset{j}{L}_{h_1}(u) \tag{8}$$

which satisfies

$$R_{12}(u_1 - u_2) T_1(u_1) T_2(u_2) = T_2(u_2) T_1(u_1) R_{12}(u_1 - u_2). \tag{9}$$

Under periodic boundary conditions, the trace of $T(u)$ over the auxiliary space, $\tau(u) = \mathrm{tr}_V\, T(u)$, gives the commuting transfer matrix depending on the spectral parameter u:

$$[\tau(u_1), \tau(u_2)] = 0. \tag{10}$$

Under open boundary conditions, the integrable system is given by recalling the reflection equation (RE) besides the YBE:

$$R_{12}(u_1 - u_2) K_1^-(u_1) R_{21}(u_1 + u_2) K_2^-(u_2) = K_2^-(u_2) R_{12}(u_1 + u_2) K_1^-(u_1) R_{21}(u_1 - u_2) \tag{11}$$

and the dual RE

$$\begin{aligned} R_{12}(-u_1 + u_2) K_1^+(u_1) R_{21}(-u_1 - u_2 - nw) K_2^+(u_2) \\ = K_2^+(u_2) R_{12}(-u_1 - u_2 - nw) K_1^+(u_1) R_{21}(-u_1 + u_2) \end{aligned} \tag{12}$$

which was proposed by Cherednik [13] and Sklyanin [14]. Here $K^-(u)$ and $K^+(u)$ are the reflection matrix and dual reflection matrix respectively. Comparing (11) with (12), we can see that there is an isomorphism between $K^+(u)$ and $K_-(u)$:

$$K^-(u) \to K^+(u) = K^-(-u - \tfrac{1}{2}nw).$$

From (1) and (11), the reflection matrix $K_-(u)$ of the Z_n Belavin model is given by [9, 20],

$$\begin{aligned} &K(u) = K_0(u)K_0(0) \qquad K_0(u) \equiv \frac{1}{n}\sum_{\alpha\in G_n} \bar{W}_{2\alpha}(u,c)\omega^{2\alpha_1\alpha_2} I_{2\alpha} \\ &\bar{W}_{2\alpha}(u,c) \equiv \frac{\sigma_{2\alpha}(u+c)}{\sigma_{2\alpha}(c)} \qquad c \text{ arbitrary parameter.} \end{aligned} \tag{13}$$

Define $\mathcal{T}(u) = T(u)K^-(u)T^{-1}(-u)$. Using equations (11) and (9) and

$$R_{12}(u_1-u_2)T_2^{-1}(u_2)T_1^{-1}(u_1) = T_1^{-1}(u_1)T_2^{-1}(u_2)R_{12}(u_1-u_2) \tag{14}$$

where $T^{-1}(u)T(u) = I \otimes \mathrm{id}$, we can prove that $\mathcal{T}(u)$ also satisfies the reflection equation

$$R_{12}(u_1-u_2)\mathcal{T}_1(u_1)R_{21}(u_1+u_2)\mathcal{T}_2(u_2) = \mathcal{T}_2(u_2)R_{12}(u_1+u_2)\mathcal{T}_1(u_1)R_{21}(u_1-u_2). \tag{15}$$

Sklyanin [14] has shown that the transfer matrix defined by

$$\begin{aligned} t(u) &= \mathrm{tr}\, K^+(u)\mathcal{T}(u) \\ &= \mathrm{tr}\, K^+(u)T(u)K^-(u)T^{-1}(-u) \end{aligned} \tag{16}$$

forms a one-parameter commutative family

$$[t(u_1), t(u_2)] = 0.$$

3. Antisymmetric fusion

Using the fusion procedure, some new integrable models may be generated based on the elementary model [40, 41]. For the Z_n Belavin model, the antisymmetrically fused R-matrices are defined as

$$R_{\langle l1\rangle\bar{a}}(u) = \prod_{j=1}^{l} R_{l-j+1,\bar{a}}(u+lw-jw)P_l^- \tag{17}$$

$$R_{a\langle\bar{l}\bar{1}\rangle}(u) = \prod_{j=1}^{l} R_{a\bar{j}}(u+w-jw)P_l^- \tag{18}$$

$$R_{a\langle\bar{1}\bar{l}\rangle}(u) = \prod_{j=1}^{l} R_{a,\overline{l-j+1}}(u-jw+lw)P_l^- \tag{19}$$

$$R_{\langle 1l\rangle\bar{a}}(u) = \prod_{j=1}^{l} R_{j\bar{a}}(u+w-jw)P_l^- \tag{20}$$

which are matrices of the space $V_1 \otimes \cdots \otimes V_l \otimes V_a$. P_l^- is the antisymmetric projector operator on the space $V^{\otimes l}$

$$\begin{aligned} &P_l^- = \frac{1}{l!}\sum_{p\in S_l} \mathrm{sgn}(p)\hat{p} \\ &\hat{p}(e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_l}) = e_{i_{p(1)}}\otimes e_{i_{p(2)}}\otimes\cdots\otimes e_{i_{p(l)}}. \end{aligned} \tag{21}$$

S_l denote the symmetric group realized as the group of permutations p of the set $\{1, 2, \ldots, l\}$, and

$$\mathrm{sgn}(p) = \begin{cases} 1 & p \text{ is even permutation} \\ -1 & p \text{ is odd permutation.} \end{cases} \tag{22}$$

Hereafter, the order of the operators after $\prod_i$ is understood to be

$$\prod_{i=1}^{k} A_{f(i)} = A_{f(1)} A_{f(2)} \ldots A_{f(k)}.$$

We can also fuse the $R_{\langle l1\rangle a}(u)$ as follows to obtain matrices of the space $V_1 \otimes \cdots \otimes V_l \otimes V_{\bar{1}} \otimes \cdots \otimes V_{\bar{m}}$:

$$\begin{aligned}
R_{\langle l1\rangle\langle\bar{m}\bar{1}\rangle}(u) &= \prod_{j=1}^{m} R_{\langle l1\rangle\bar{j}}(u - jw + w)P_{\bar{m}}^{-} = \prod_{i=1}^{l} R_{l-i+1,\langle\bar{m}\bar{1}\rangle}(u + lw - jw)P_l^{-} \\
R_{\langle 1l\rangle\langle\bar{1}\bar{m}\rangle}(u) &= \prod_{j=1}^{m} R_{\langle 1l\rangle\overline{m-j+1}}(u + mw - jw)P_{\bar{m}}^{-} = \prod_{i=1}^{l} R_{i\langle\bar{1}\bar{m}\rangle}(u - iw + w)P_l^{-} \\
R_{\langle 1l\rangle\langle\bar{m}\bar{1}\rangle}(u) &= \prod_{j=1}^{m} R_{\langle 1l\rangle\bar{j}}(u - jw + w)P_{\bar{m}}^{-} = \prod_{i=1}^{l} R_{i\langle\bar{m}\bar{1}\rangle}(u - iw + w)P_l^{-} \\
R_{\langle l1\rangle\langle\bar{1}\bar{m}\rangle}(u) &= \prod_{j=1}^{m} R_{\langle l1\rangle\overline{m-j+1}}(u + jw - w)P_{\bar{m}}^{-} = \prod_{i=1}^{m} R_{l-i+1,\langle\bar{1}\bar{m}\rangle}(u + iw - w)P_l^{-}.
\end{aligned} \tag{23}$$

These fused R-matrices satisfy the fused YBE

$$R_{\langle ab\rangle\langle\bar{c}\bar{d}\rangle} R_{\langle ab\rangle\langle\bar{p}\bar{q}\rangle} R_{\langle cd\rangle\langle\bar{p}\bar{q}\rangle} = R_{\langle cd\rangle\langle\bar{p}\bar{q}\rangle} R_{\langle ab\rangle\langle\bar{p}\bar{q}\rangle} R_{\langle ab\rangle\langle\bar{c}\bar{d}\rangle} \tag{24}$$

the fused unitarity

$$R_{\langle l1\rangle\langle\bar{m}\bar{1}\rangle}(u) R_{\langle\bar{m}\bar{1}\rangle\langle l1\rangle}(-u) = g(u - v)P_l^{-} \otimes P_m^{-} \tag{25}$$

$$R_{\langle\bar{1}\bar{m}\rangle\langle 1l\rangle}(u) R_{\langle 1l\rangle\langle\bar{1}\bar{m}\rangle}(-u) = g(u - v)P_l^{-} \otimes P_{\bar{m}}^{-} \tag{26}$$

and the fused crossing unitarity

$$R^{t_{\bar{1}}\ldots t_{\bar{m}}}_{\langle\bar{1}\bar{m}\rangle\langle l1\rangle}(-u - nw) R^{t_{\bar{1}}\ldots t_{\bar{m}}}_{\langle l1\rangle\langle\bar{1}\bar{m}\rangle}(u) = \tilde{g}(u)P_{\bar{m}}^{-} \otimes P_l^{-} \tag{27}$$

$$R^{t_1\ldots t_l}_{\langle 1l\rangle\langle\bar{m}\bar{1}\rangle}(-u - nw) R^{t_1\ldots t_l}_{\langle\bar{m}\bar{1}\rangle\langle 1l\rangle}(u) = \tilde{g}(u)P_{\bar{m}}^{-} \otimes P_l^{-} \tag{28}$$

here, $g(u) = \prod_{i=1}^{l}\prod_{j=1}^{m} \rho(u + iw - jw)$, and $\tilde{g}(u) = \prod_{i=1}^{l}\prod_{j=1}^{m} \tilde{\rho}(u + iw + jw - 2w)$.

The fused $T(u)$ are defined as

$$\begin{aligned}
T_{\langle l1\rangle}(u) &= \prod_{i=1}^{l} T_{l-i+1}(u + lw - iw)P_l^{-} \\
T^{-1}_{\langle l1\rangle}(-u) &= \prod_{i=1}^{l} T^{-1}_{l-i+1}(-u - lw + iw)P_l^{-}
\end{aligned} \tag{29}$$

where P_l^{-} is the antisymmetric projector of the auxiliary spaces. The fused $T_{\langle l1\rangle}(u)(T^{-1}_{\langle l1\rangle}(-u))$ satisty the following fused Yang–Baxter relation in view of (9),

$$\begin{aligned}
&R_{\langle\bar{m}\bar{1}\rangle\langle l1\rangle}(v - u)T_{\langle\bar{m}\bar{1}\rangle}(v)T_{\langle l1\rangle}(u) = T_{\langle l1\rangle}(u)T_{\langle\bar{m}\bar{1}\rangle}(v)R_{\langle\bar{m}\bar{1}\rangle\langle l1\rangle}(v - u) \\
&R_{\langle\bar{1}\bar{m}\rangle\langle 1l\rangle}(u - v)T^{-1}_{\langle l1\rangle}(-u)T^{-1}_{\langle\bar{m}\bar{1}\rangle}(-v) = T^{-1}_{\langle\bar{m}\bar{1}\rangle}(-v)T^{-1}_{\langle l1\rangle}(-u)R_{\langle\bar{1}\bar{m}\rangle\langle 1l\rangle}(u - v) \\
&T^{-1}_{\langle\bar{m}\bar{1}\rangle}(-v)R_{\langle l1\rangle\langle\bar{1}\bar{m}\rangle}(u + v)T_{\langle l1\rangle}(u) = T_{\langle l1\rangle}(u)R_{\langle l1\rangle\langle\bar{1}\bar{m}\rangle}(u + v)T^{-1}_{\langle\bar{m}\bar{1}\rangle}(-v).
\end{aligned} \tag{30}$$

For boundary matrices $K^-(u)$ and $K^+(u)$, we define the following fused boundary matrices:

$$K^-_{\langle l1\rangle}(u) = \prod_{i=1}^{l}\left[\prod_{j=1}^{i-1} R_{l-i+1,l-j+1}(2u+(2l-i-j)w)\right]K^-_{l-i+1}(u+(l-i)w)P^-_l$$
$$K^+_{\langle l1\rangle}(u) = \prod_{i=1}^{l} K^+_i(u+(i-1)w)\left[\prod_{j=1}^{l-i} R_{i+j,i}(-2u-(2i+j-2)w-nw)\right]P^-_l. \tag{31}$$

Now, we will prove that $K^-_{\langle l1\rangle}(u)$ and $K^+_{\langle l1\rangle}(u)$ satisfy

$$K^-_{\langle l1\rangle}(u) = P^-_l K^-_{\langle l1\rangle}(u) \tag{32}$$
$$K^+_{\langle l1\rangle}(u) = P^-_l K^+_{\langle l1\rangle}(u). \tag{33}$$

First from the fact [40, 41] that $R_{12}(-w) = P^-_{12}A$, here $P^-_{12} = \frac{1}{2}(1-P_{12})$, A is an invertible matrix, we have

$$R_{l\ldots 21}(-w) \equiv \prod_{j=1}^{l-1}\prod_{i=j+1}^{l} R_{ij}(-(i-j)w)$$
$$= \prod_{j=1}^{l-1} R_{\langle j+1,l\rangle j}(-w) = P^-_l B'. \tag{34}$$

B' is an invertible matrix. Then

$$\begin{aligned}
\text{LHS of (32)} = &\prod_{i=1}^{l-2} R_{l-i+1,\langle l-i+2,l\rangle}(2u+(2l-2i+1)w)K^-_{l-i+1}(u+(l-i)w)\\
&\times R_{2\langle 3l\rangle}(2u+3w)K^-_2(u+w)R_{1\langle 3l\rangle}(2u+2w)R_{12}(2u+w)K^-_1(u)\\
&\times R_{21}(-w)R_{\langle 3l\rangle 1}(-2w)\prod_{j=2}^{l-1} R_{\langle j+1,l\rangle j}(-w)B'^{-1}\\
= &\prod_{i=1}^{l-2} R_{l-i+1,\langle l-i+2,l\rangle}(2u+(2l-2i+1)w)K^-_{l-i+1}(u+(l-i)w)\\
&\times R_{2\langle 3l\rangle}(2u+3w)R_{1\langle 3l\rangle}(2u+2w)K^-_2(u+w)R_{12}(2u+w)K^-_1(u)\\
&\times R_{21}(-w)R_{\langle 3l\rangle 1}(-2w)\prod_{j=2}^{l-1} R_{\langle j+1,l\rangle j}(-w)B'^{-1}.
\end{aligned}$$

Using RE(11), we have

$$\begin{aligned}
\text{LHS of (32)} = &\prod_{i=1}^{l-2} R_{l-i+1,\langle l-i+2,l\rangle}(2u+(2l-2i+1)w)K^-_{l-i+1}(u+(l-i)w)\\
&\times R_{2\langle 3l\rangle}(2u+3w)R_{1\langle 3l\rangle}(2u+2w)R_{12}(-w)K^-_1(u)R_{21}(2u+w)\\
&\times K^-_2(u+w)R_{\langle 3l\rangle 1}(-2w)\prod_{j=2}^{l-1} R_{\langle j+1,l\rangle j}(-w)B'^{-1}.
\end{aligned}$$

By YBE (24) this becomes

$$\text{LHS of (32)} = R_{12}(-w)\prod_{i=1}^{l-2} R_{l-i+1,\langle l-i+2,l\rangle}(2u+(2l-2i+1)w)K^-_{l-i+1}(u+(l-i)w)$$

$$\times R_{1\langle 3l\rangle}(2u+2w)K_1^-(u)R_{2\langle 3l\rangle}(2u+3w)R_{21}(2u+w)$$
$$\times K_2^-(u+w)R_{\langle 3l\rangle 1}(-2w)\prod_{j=2}^{l-1}R_{\langle j+1,l\rangle j}(-w)B'^{-1}.$$

In this way, we can move the $K_1^-(u)$ to the left of the other $K_j^-(u)$ $(j\neq 1)$ one by one, and obtain

$$\begin{aligned}\text{LHS of (32)} &= R_{1\langle l2\rangle}(-w)K_1^-(u)R_{\langle l2\rangle 1}(w)\\ &\quad\times\prod_{i=1}^{l-1}R_{l-i+1,\langle l-i+2,l\rangle}(2u+(2l-2i+1)w)K_{l-i+1}^-(u+(l-i)w)\\ &\quad\times\prod_{j=2}^{l-1}R_{\langle j+1,l\rangle j}(-w)B'^{-1}.\end{aligned}$$

Repeating this procedure, we have

$$\begin{aligned}\text{LHS of (32)} &= \prod_{i=1}^{l}R_{l-i+1,\langle l-i+2,l\rangle}(-w)\prod_{j=1}^{l}K_j^-(u+(j-1)w)R_{\langle j+1,l\rangle j}(2u+(2j-1)w)\\ &= P_l^-\prod_{i=1}^{l}R_{l-i+1,\langle l-i+2,l\rangle}(-w)\\ &\quad\times\prod_{j=1}^{l}K_j^-(u+(j-1)w)R_{\langle j+1,l\rangle j}(2u+(2j-1)w).\end{aligned}$$

Here, we use

$$\begin{aligned}\prod_{i=1}^{l}R_{l-i+1,\langle l-i+2,l\rangle}(-w) = R_{l\ldots 1}(-w) &= P_l^-B'\\ &= P_l^-R_{l\ldots 1}(-w)\end{aligned}$$

and obtain

$$\begin{aligned}\text{LHS of (32)} &= P_l^-\prod_{i=1}^{l}R_{l-i+1,\langle l-i+2,l\rangle}(2u+(2l-2i+1)w)K_{l-i+1}^-(u+(l-i)w)P_l^-\\ &= \text{RHS of (32)}.\end{aligned}$$

Then a tedious but straightforward calculation similar to that used in the proof of equation (32) leads to the following fused RE

$$\begin{aligned}&R_{\langle\bar m\bar 1\rangle\langle l1\rangle}(v-u)K_{\langle\bar m\bar 1\rangle}^-(v)R_{\langle l1\rangle\langle\bar 1\bar m\rangle}(u+v)K_{\langle l1\rangle}^-(u)\\ &\quad= K_{\langle l1\rangle}^-(u)R_{\langle\bar m\bar 1\rangle\langle 1l\rangle}(u+v)K_{\langle\bar m\bar 1\rangle}^-(v)R_{\langle 1l\rangle\langle\bar 1\bar m\rangle}(v-u)\end{aligned}\tag{35}$$

and fused dual RE

$$\begin{aligned}&K_{\langle l1\rangle}^+(u)R_{\langle\bar 1\bar m\rangle\langle l1\rangle}(-u-v-nw)K_{\langle\bar m\bar 1\rangle}^+(v)R_{\langle l1\rangle\langle\bar m\bar 1\rangle}(u-v)\\ &\quad= R_{\langle\bar 1\bar m\rangle\langle 1l\rangle}(u-v)K_{\langle\bar m\bar 1\rangle}^+(v)R_{\langle 1l\rangle\langle\bar m\bar 1\rangle}(-u-v-nw)K_{\langle l1\rangle}^+(u).\end{aligned}\tag{36}$$

Define $\mathcal{T}_{\langle l1\rangle}(u)\equiv T_{\langle l1\rangle}(u)K_{\langle l1\rangle}^-(u)T_{\langle l1\rangle}^{-1}(-u)$, equations (30) and (35), it is straightforward to show

$$\begin{aligned}&R_{\langle m1\rangle\langle l1\rangle}(v-u)\mathcal{T}_{\langle m1\rangle}(v)R_{\langle l1\rangle\langle 1m\rangle}(u+v)\mathcal{T}_{\langle l1\rangle}(u)\\ &\quad= \mathcal{T}_{\langle l1\rangle}(u)R_{\langle m1\rangle\langle 1l\rangle}(u+v)\mathcal{T}_{\langle m1\rangle}(v)R_{\langle 1l\rangle\langle 1m\rangle}(v-u)\end{aligned}\tag{37}$$

which is similar to equation (35). When $l = n$, the Sklyanin determinant $S\det\mathcal{T}(u)$ for operator $\mathcal{T}_{\langle n1\rangle}(u)$ is defined as

$$S\det\mathcal{T}(u) = \mathcal{T}_{\langle n1\rangle}(u) = P_n^-\mathcal{T}_{\langle n1\rangle}(u) \tag{38}$$

which is the algebraic centre of the operator algebra $\mathcal{T}(u)$. That is

$$[\mathcal{T}(v), S\det\mathcal{T}(u)] = 0. \tag{39}$$

To prove this, first we must show

$$R_{\bar{1}\langle n1\rangle}(u) = \xi(u)I\otimes P_n^- \tag{40}$$

here $\xi(u)$ is a scalar function depending on u. Indeed, recall that P_n^- is a one-dimensional projection in $V^{\otimes n}$, $R_{\bar{1}\langle n1\rangle}(u)$ may be written as $U\otimes P_n^-$ acting on $V\otimes V^{\otimes n}$. On the other hand, due to the Z_n symmetry of equation (2), we have

$$\begin{aligned}
I_\alpha\otimes I^{\otimes n}R_{\bar{1}\langle n1\rangle}(u)I_\alpha^{-1}\otimes I^{\otimes n} &= P_n^- I\otimes(I_\alpha^{-1})^{\otimes n}\prod_{j=1}^{n}R_{\bar{1}j}(u+(1-j)w)I\otimes I_\alpha^{\otimes n}P_n^- \\
&= P_n^-\prod_{j=1}^{n}R_{\bar{1}j}(u+(1-j)w)P_n^- \\
&= R_{\bar{1}\langle n1\rangle}.
\end{aligned} \tag{41}$$

So $U = \xi(u)I$. In the same way, one can write

$$\begin{aligned}
R_{\langle 1n\rangle\bar{1}}(u) &= \xi(u)P_n^-\otimes I \\
R_{\langle n1\rangle\bar{1}}(u) &= \eta(u)P_n^-\otimes I \\
R_{\bar{1}\langle 1n\rangle}(u) &= \eta(u)I\otimes P_n^-.
\end{aligned}$$

Here, $\eta(u)$ is also a scalar function depending on u. Now let $m = 1$ and $l = n$ as in equation (37), we have

$$\xi(v-u)P_n^-\mathcal{T}_{\bar{1}}(v)\eta(u+v)P_n^-\mathcal{T}_{\langle n1\rangle}(u) = \mathcal{T}_{\langle n1\rangle}(u)\eta(u+v)P_n^-\mathcal{T}_{\bar{1}}(v)\xi(v-u)P_n^-. \tag{42}$$

So $\mathcal{T}_{\bar{1}}(v)P_n^-\mathcal{T}_{\langle n1\rangle}(u) = P_n^-\mathcal{T}_{\langle n1\rangle}(u)\mathcal{T}_{\bar{1}}(v)$ which proves (39).

4. Commuting family

In this section, we shall show the fused operators

$$\begin{aligned}
t_m(u) &\equiv \mathrm{tr}_{1\ldots m}\, K^+_{\langle m1\rangle}(u)T_{\langle m1\rangle}(u)K^-_{\langle m1\rangle}(u)T^{-1}_{\langle m1\rangle}(-u) \\
&= \mathrm{tr}_{1\ldots m}\, K^+_{\langle m1\rangle}(u)\mathcal{T}_{\langle m1\rangle}(u) \qquad (1\leqslant m\leqslant n)
\end{aligned} \tag{43}$$

form a commuting family.

Given $1\leqslant m, l\leqslant n$,

$$\begin{aligned}
t_m(v)t_l(u) &= \mathrm{tr}_{\bar{1}\ldots\bar{m}}\, K^{+t_{\bar{m}}\ldots t_{\bar{1}}}_{\langle\bar{m}\bar{1}\rangle}(v)\mathcal{T}^{t_{\bar{m}}\ldots t_{\bar{1}}}_{\langle\bar{m}\bar{1}\rangle}(u)tr_{1\ldots l}K^+_{\langle l1\rangle}(u)\mathcal{T}_{\langle l1\rangle}(u) \\
&= \mathrm{tr}_{\bar{1}\ldots\bar{m}}\,\mathrm{tr}_{1\ldots l}\, K^{+t_{\bar{m}}\ldots t_{\bar{1}}}_{\langle\bar{m}\bar{1}\rangle}(v)K^+_{\langle l1\rangle}(u)\mathcal{T}^{t_{\bar{m}}\ldots t_{\bar{1}}}_{\langle\bar{m}\bar{1}\rangle}\mathcal{T}_{\langle l1\rangle}(u)
\end{aligned}$$

inserting the cross-unitarity (27)

$$\begin{aligned}
t_m(v)t_l(u) = {} & \frac{1}{\tilde{g}_{ml}(u+v)}\,\mathrm{tr}_{\bar{1}\ldots\bar{m}}\,\mathrm{tr}_{1\ldots l}\, K^{+t_{\bar{m}}\ldots t_{\bar{1}}}_{\langle\bar{m}\bar{1}\rangle}(v)K^+_{\langle l1\rangle}(u) \\
& \times R^{t_{\bar{1}}\ldots t_{\bar{m}}}_{\langle\bar{1}\bar{m}\rangle\langle l1\rangle}(-u-v-nw)R^{t_{\bar{1}}\ldots t_{\bar{m}}}_{\langle l1\rangle\langle\bar{1}\bar{m}\rangle}(u+v)\mathcal{T}^{t_{\bar{m}}\ldots t_{\bar{1}}}_{\langle\bar{m}\bar{1}\rangle}(v)\mathcal{T}_{\langle l1\rangle}(u) \\
= {} & \frac{1}{\tilde{g}_{ml}(u+v)}\,\mathrm{tr}_{\bar{1}\ldots\bar{m}}\,\mathrm{tr}_{1\ldots l}[K^+_{\langle l1\rangle}(u)R_{\langle\bar{1}\bar{m}\rangle\langle l1\rangle}(-u-v-nw)K^+_{\langle\bar{m}\bar{1}\rangle}(v)]^{t_{\bar{1}}\ldots t_{\bar{m}}} \\
& \times[\mathcal{T}_{\langle\bar{m}\bar{1}\rangle}(v)R_{\langle l1\rangle\langle\bar{1}\bar{m}\rangle}(u+v)\mathcal{T}_{\langle l1\rangle}(u)]^{t_{\bar{1}}\ldots t_{\bar{m}}}.
\end{aligned}$$

Due to $\mathrm{tr}(A^t B^t) = \mathrm{tr}(AB)$, we have

$$t_m(v)t_l(u) = \frac{1}{\tilde{g}_{ml}(u+v)} \mathrm{tr}_{\bar{1}\ldots\bar{m}} \mathrm{tr}_{1\ldots l}[K^+_{\langle l1\rangle}(u) R_{\langle\bar{1}\bar{m}\rangle\langle l1\rangle}(-u-v-nw) K^+_{\langle\bar{m}\bar{1}\rangle}(v) \\ \times \mathcal{T}_{\langle\bar{m}\bar{1}\rangle}(v) R_{\langle l1\rangle\langle\bar{1}\bar{m}\rangle}(u+v) \mathcal{T}_{\langle l1\rangle}(u)].$$

Inserting the unitarity (25), we have

$$t_m(v)t_l(u) = \frac{1}{\tilde{g}_{ml}(u+v) g_{ml}(u-v)} \mathrm{tr}_{\bar{1}\ldots\bar{m}} \mathrm{tr}_{1\ldots l}[K^+_{\langle l1\rangle}(u) R_{\langle\bar{1}\bar{m}\rangle\langle l1\rangle}(-u-v-nw) K^+_{\langle\bar{m}\bar{1}\rangle}(v) \\ \times R_{\langle l1\rangle\langle\bar{m}\bar{1}\rangle}(u-v) R_{\langle\bar{m}\bar{1}\rangle\langle l1\rangle}(v-u) \mathcal{T}_{\langle\bar{m}\bar{1}\rangle}(v) R_{\langle l1\rangle\langle\bar{1}\bar{m}\rangle}(u+v) \mathcal{T}_{\langle l1\rangle}(u)].$$

Applying the fused RE (36) and (37) this becomes

$$t_m(v)t_l(u) = \frac{1}{\tilde{g}_{ml}(u+v) g_{ml}(u-v)} \mathrm{tr}_{\bar{1}\ldots\bar{m}} \mathrm{tr}_{1\ldots l}[R_{\langle\bar{1}\bar{m}\rangle\langle 1l\rangle}(u-v) K^+_{\langle\bar{m}\bar{1}\rangle}(v) \\ \times R_{\langle 1l\rangle\langle\bar{m}\bar{1}\rangle}(-u-v-nw) K^+_{\langle l1\rangle}(u) \mathcal{T}_{\langle l1\rangle}(u) R_{\langle\bar{m}\bar{1}\rangle\langle 1l\rangle}(u+v) \\ \times \mathcal{T}_{\langle\bar{m}\bar{1}\rangle}(v) R_{\langle 1l\rangle\langle\bar{1}\bar{m}\rangle}(v-u)].$$

With the help of unitarity (26), this becomes

$$t_m(v)t_l(u) = \frac{1}{\tilde{g}_{ml}(u+v)} \mathrm{tr}_{\bar{1}\ldots\bar{m}} \mathrm{tr}_{1\ldots l}[K^+_{\langle\bar{m}\bar{1}\rangle}(v) R_{\langle 1l\rangle\langle\bar{m}\bar{1}\rangle}(-u-v-nw) K^+_{\langle l1\rangle}(u) \\ \times \mathcal{T}_{\langle l1\rangle}(u) R_{\langle\bar{m}\bar{1}\rangle\langle 1l\rangle}(u+v) \mathcal{T}_{\langle\bar{m}\bar{1}\rangle}(v)].$$

Apply the transposition $t_1 \ldots t_l$, and use cross-unitarity (28), we finally have

$$\begin{aligned} t_m(v)t_l(u) &= \frac{1}{\tilde{g}_{ml}(u+v)} \mathrm{tr}_{\bar{1}\ldots\bar{m}} \mathrm{tr}_{1\ldots l}[K^+_{\langle\bar{m}\bar{1}\rangle}(v) K^{+t_1\ldots t_l}_{\langle l1\rangle}(u) R^{t_1\ldots t_l}_{\langle 1l\rangle\langle\bar{m}\bar{1}\rangle}(-u-v-nw) \\ &\qquad \times R^{t_1\ldots t_l}_{\langle\bar{m}\bar{1}\rangle\langle 1l\rangle}(u+v) \mathcal{T}^{t_1\ldots t_l}_{\langle l1\rangle}(u) \mathcal{T}_{\langle\bar{m}\bar{1}\rangle}(v)] \\ &= \mathrm{tr}_{\bar{1}\ldots\bar{m}} \mathrm{tr}_{1\ldots l}[K^+_{\langle\bar{m}\bar{1}\rangle}(v) K^{+t_1\ldots t_l}_{\langle l1\rangle}(u) \mathcal{T}^{t_1\ldots t_l}_{\langle l1\rangle}(u) \mathcal{T}_{\langle\bar{m}\bar{1}\rangle}(v)] \\ &= \mathrm{tr}_{1\ldots l} K^{+t_l\ldots t_1}_{\langle l1\rangle}(u) \mathcal{T}^{t_l\ldots t_1}_{\langle l1\rangle}(u) \, \mathrm{tr}_{\bar{1}\ldots\bar{m}} K^+_{\langle\bar{m}\bar{1}\rangle}(v) \mathcal{T}_{\langle\bar{m}\bar{1}\rangle}(v) \\ &= t_l(u) t_m(v). \end{aligned}$$

5. Difference operator expression

To relate the commuting operators t_m with the difference operators, we use the factorized L operator in [23, 31], which is involved in the intertwiner of face–vertex model. The details can be found in [21–26].

Let $a, b \in Z^{\otimes n}$, the bases of $Z^{\otimes n}$, consist of

$$e_i = (0, \ldots 0, 1, 0, \ldots 0) \qquad 0 \leqslant i \leqslant n-1. \tag{44}$$

We call (a, b) an admissible pair if $a - b = e_i$. For the Z_n Belavin model, Jimbo *et al* (1988) introduced the following intertwiner

$$\begin{aligned} &\phi^b_a(u) = (\phi^b_{a0}(u), \phi^b_{a1}(u), \ldots \phi^b_{an-1}(u))^{\mathrm{t}} \\ &a = (m_0, \ldots, m_{n-1}). \end{aligned} \tag{45}$$

Here t means transposition, and

$$\begin{aligned} &\phi^b_{ai}(u) \equiv \begin{cases} \theta^{(i)}(u + nw\bar{m}_k) & \text{if } a - b = e_k \\ 0 & \text{otherwise} \end{cases} \\ &\theta^{(i)}(u) = \theta \begin{bmatrix} \frac{1}{2} - \frac{i}{n} \\ \frac{1}{2} \end{bmatrix} (u, n\tau) \end{aligned} \tag{46}$$

where $\bar{m}_i = m_i - \frac{1}{n}\sum_l m_l + \lambda_i$, and $\{\lambda_i\}$ is a set of generic numbers. According to Bazhanov *et al* [21, 42], one can find two vectors

$$\begin{aligned}&\bar{\phi}^b_a(u) = (\bar{\phi}^{b0}_a(u), \ldots, \bar{\phi}^{bn-1}_a(u))\\ &\tilde{\phi}^b_a(u) = (\tilde{\phi}^{b0}_a(u), \ldots, \tilde{\phi}^{bn-1}_a(u)) \qquad \text{if } (ab) \text{ is admissible}\end{aligned} \tag{47}$$

such that

$$\sum_i \bar{\phi}^{a-e_k,i}_a(u)\phi^{a-e_j,i}_a(u) = \delta_{jk}$$
$$\sum_i \bar{\phi}^{a-e_i,j}_a(u)\phi^{a-e_i,k}_a(u) = \delta_{jk} \tag{48}$$
$$\sum_i \tilde{\phi}^{a,i}_{a+e_k}(u)\phi^a_{a+e_j,i}(u) = \delta_{jk}$$
$$\sum_i \tilde{\phi}^{a,j}_{a+e_i}(u)\phi^a_{a+e_i,k}(u) = \delta_{jk}. \tag{49}$$

Define $L^j_i(u) \equiv \sum_k \Gamma_k \phi^a_{a+e_k,i}(u+\delta)\tilde{\phi}^{a,j}_{a+e_k}(u)$, where the difference operator Γ_k is defined as

$$\Gamma_k f(a) = f(a - e_k)\Gamma_k. \tag{50}$$

One can show that $L^j_i(u)$ satisfies equation (7). The inverse operators of $L^j_i(u)$ are defined by

$$L^{-1}(\delta|u)^j_i = \sum_{k=1}^n \Gamma_{-k}\phi^{a-e_k}_{a,i}(u)\bar{\phi}^{a-e_k,j}_a(u+\delta).$$

Utilizing equations (48)–(49), it is straightforward to prove

$$L^-(\delta|u)L(\delta|u) = L(\delta|u)L^-(\delta|u) = \mathrm{id}$$

and

$$\begin{aligned}&\sum_{k,l} R^{kl}_{ij}(u_2-u_1)\overset{2}{L}{}^{-1}(\delta|-u_2)^{j'}_l \overset{1}{L}{}^{-1}(\delta|-u_1)^{i'}_k\\ &\quad = \sum_{k,l} \overset{1}{L}{}^{-1}(\delta|-u_1)^k_i \overset{2}{L}{}^{-1}(\delta|-u_2)^l_j R^{i'j'}_{kl}(u_2-u_1).\end{aligned} \tag{51}$$

Let $f(a)$ be a function on $Z^{\otimes n}$, we define the actions of L and L^{-1} operators on $f(a)$ as

$$\begin{aligned}(L(\delta|u)^j_i f)(a) &= \sum_{k=1}^n \Gamma_k \phi^a_{a+e_k,i}(u+\delta)\tilde{\phi}^{a,j}_{a+e_k}(u) f(a)\\ &= \sum_{k=1}^n \phi^{a-e_k}_{a,i}(u+\delta)\tilde{\phi}^{a-e_k,j}_a(u) f(a+e_k)\\ (L^{-1}(\delta|u)^j_i f)(a) &= \sum_{k=1}^n \Gamma_{-k}\phi^{a-e_k}_{a,i}(u)\bar{\phi}^{a-e_k,j}_a(u+\delta) f(a)\\ &= \sum_{k=1}^n \phi^a_{a+e_k,i}(u)\bar{\phi}^{a,j}_{a+e_k}(u+\delta) f(a-e_k).\end{aligned} \tag{52}$$

Let $T(u) = L(u)$, from the definition of t_m (43), the action of t_m on $f(a)$ is

$$(t_m f)(a) = \sum_{i,j,k,l} K^{+}_{\langle m1\rangle}(u)^{l_1\ldots l_m}_{i_1\ldots i_m} K^{-}_{\langle m1\rangle}(u)^{j_1\ldots j_m}_{k_1\ldots k_m} L^{-1}(\delta| - u - (m-1)w)^{i_m}_{j_m} \ldots L^{-1}(\delta| - u)^{i_1}_{j_1} L(\delta|u + (m-1)w)^{k_m}_{l_m} \ldots L(\delta|u)^{k_1}_{l_1} f(a). \tag{53}$$

This set of difference operators $\{t_m\}$ forms a commutative family of the Macdonald type.

6. The Hamiltonian of integrable model

Any difference operator of the commuting family $\{t_m\}$ can be chosen as the Hamiltonian of an integrable model with the conservative family $\{t_m\}$. This is similar to the RS [34] model. In fact, if we use periodic boundary conditions for the Z_n Belavin model, we can obtain the RS model and the degenerate form gives the Calogero–Moser (CM) model. Let us discuss this in more detail with open boundary conditions. We select t_1, which is the simplest in form, as is the Hamiltonian. For the reflection matrix given in (13), after some lengthy but straightforward calculation we have, for $\delta = 0$,

$$t_1(u) = \sum_{\mu\nu} \Gamma_{-\mu}\Gamma_\nu F^{(1)}_{\mu\nu}(a,u) F^{(2)}_{\mu\nu}(a,u) \tag{54}$$

where

$$F^{(1)}_{\mu\nu}(a,u) = \frac{1}{n}\sum_\gamma \bar{W}_{2\gamma}(u,c) A_{2\gamma}$$

$$F^{(2)}_{\mu\nu}(a,u) = \frac{1}{n}\sum_\gamma \bar{W}_{2\gamma}\left(-u - \frac{nw}{2}, c\right) B_{2\gamma}$$

$$\begin{aligned}
A_{2\gamma}(a,u)_{\mu\nu} &= \exp\left(-2\mathrm{i}\pi\frac{2\gamma_1}{n}(\gamma_2 - 1)\right) \\
&\quad\times \frac{\sigma_{(2\gamma_1,2\gamma_2+1)}[-u + w(2-n-\eta) + \frac{n-3}{2} + w(\bar{a}_\mu + \bar{a}_\nu - \delta_{\mu\nu})]}{\sigma_0[u + w\eta - w(1-n) - \frac{n-1}{2}]} \\
&\quad\times \prod_{j\neq\nu} \frac{\sigma_{(2\gamma_1,2\gamma_2+1)}[w(\bar{a}_j + \bar{a}_\mu - \delta_{\mu j} + 1)]}{\sigma_0[w(\bar{a}_j - \bar{a}_\nu - \delta_{\mu j} + \delta_{\mu\nu})]}
\end{aligned} \tag{55}$$

$$\begin{aligned}
B_{2\gamma}(a,u)_{\mu\nu} &= \exp\left(-2\mathrm{i}\pi\frac{2\gamma_1}{n}(\gamma_2 - 1)\right) \\
&\quad\times \frac{\sigma_{(2\gamma_1,2\gamma_2+1)}[u - w\eta + \frac{n-3}{2} + w(\bar{a}_\mu + \bar{a}_\nu - \delta_{\mu\nu} + 1))}{\sigma_0(u - w\eta + \frac{n-1}{2n})} \\
&\quad\times \prod_{j\neq\mu} \frac{\sigma_{(2\gamma_1,2\gamma_2+1)}[w(\bar{a}_j + \bar{a}_\nu - \delta_{\mu\nu} + 1)]}{\sigma_0[w(\bar{a}_j - \bar{a}_\mu)]}
\end{aligned}$$

$$\bar{a}_j \equiv a_j - \frac{1}{n}\sum_k a_k + \lambda_k \qquad \sum_k \lambda_k = \eta.$$

If we choose $c = \epsilon\tau + c'$ in (13) ($\epsilon \ll \frac{1}{n}$) and make the spectral parameter u tend to $-\mathrm{i}\infty$, the trigonometric limit $\tau \to \mathrm{i}\infty$ gives

$$t_1(u) \to \sum_{\mu\nu} \Gamma_\nu \mathrm{e}^{\mathrm{i}\pi nw(a_\nu + \lambda_\nu)} \prod_{j\neq\nu} \frac{1}{\sin(\pi w(a_j - a_\nu))} \Gamma_{-\mu} \mathrm{e}^{-\mathrm{i}\pi nw(a_\mu + \lambda_{mu})} \prod_{k\neq\mu} \frac{1}{\sin(\pi w(a_k - a_\mu))}. \tag{56}$$

In quantum mechanics, momenta operators, $\hat{p}_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x_\mu}$, act on function $f(x) = f(x_0, \ldots, x_{n-1})$ as

$$\exp\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_\mu}\right) f(x_0, \ldots, x_{n-1}) = f\left(x_0, \ldots, x_\mu + \frac{\hbar}{\mathrm{i}}, \ldots\right) \exp\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_\mu}\right). \tag{57}$$

Comparing this with (50), we may replace $\Gamma_{-\mu}$ by $\exp(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_\mu})$, Γ_μ by $\exp(-\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_\mu})$ and $iw(a_i + \lambda_i)$ by x_i. The corresponding quantum Hamiltonian is

$$H(\hat{p}, q) = \sum_{\mu,\nu} \mathrm{e}^{-\hat{p}_\nu + \pi n x_\nu} \prod_{i:i\neq\nu} \frac{1}{\sinh[\pi(x_i - x_\nu)]} \mathrm{e}^{\hat{p}_\mu - \pi n x_\mu} \prod_{j:j\neq\mu} \frac{1}{\sinh[\pi(x_j - x_\mu)]}. \tag{58}$$

Here $\hat{p}_\nu = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q_\nu}$. This suggests an integrable system which has $\{t_m\}$ as its conservative quantities. For other reflection matrices or for taking other limits, we can also construct integrable models.

7. Discussion

In this paper, we study the antisymmetric fusion of the Z_n Belavin model with the open boundary condition. We give the R-matrices, reflection matrices K and L operators of the antisymmetric fusion hierarchies. The Sklyanin determinant, which is the centre of the operator algebra of the Z_n Belavin model with the open boundary condition, is constructed. We also prove that the transfer matrices t_m of the fusion hierarchies mutually commute. Utilizing the factorized L operators, we express the commuting transfer matrices as mutually commuting difference operators acting on a function space, which are related to Ruijsenaars–Macdonald-type operators. These commuting operators describe the symmetry of an integrable model similar to the RS–CM model.

Owing to the complication of the R- and K-matrices in t_m, equation (53) could not be simplified to a simple form as was done in the periodic boundary condition [31]. In Hasegawa [31] studied the fused Z_n Belavin model with periodic boundary conditions, extended the equivalence between Macdonald's operators and Ruijsenaars' one to the present elliptic case, namely between the commuting difference operators and relativistic elliptic CM system [34]. Our commuting difference operators are obtained from the fused Z_n Belavin model with an open boundary condition. We suggest that our results are related to the boundary relativistic elliptic CM system.

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